

On the robustness of power-law random graphs in the finite mean, infinite variance region

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Abstract

We consider a conditionally Poissonian random graph model where the mean degrees, ‘capacities’, follow a power-tailed distribution with finite mean and infinite variance. Such a graph of size N has a giant component which is super-small in the sense that the typical distance between vertices is of the order of $\log \log N$. The shortest paths travel through a core consisting of nodes with high mean degrees. In this paper we derive upper bounds of the distance between two random vertices when an upper part of the core is removed, including the case that the whole core is removed.

1 Introduction

Power-law random graphs (or ‘scale-free random graphs’) have become popular objects of both applied and theoretical interest, because they are simple to define and generate and yet share some important characteristics with many large and complex real-world networks. The mathematical models help to understand how and on what conditions those characteristics emerge, and this understanding can be valuable also in the design of new, artificial networks.

The general characteristic of power-law random graphs is that their degree distribution possesses a regularly varying tail, the most interesting region of tail exponents being that of finite mean and infinite variance. Several models with these features have been studied. In this paper, we work with the conditionally Poissonian random graph [7], a modification of the expected degree sequence model of Chung and Lu [3]. We draw an i.i.d. sequence of mean degrees, ‘capacities’, that follow a Pareto distribution with finite mean and infinite variance.

Our graph of size N has a giant component which is ultra-small in the sense that the typical distance between vertices is proportional to $\log \log N$ (more precisely, this is an upper bound, like most of our distance results), whereas a similar random graph whose degrees have the same mean but finite variance cannot offer better than $\log N$ scalability of distances. Remarkably, the speciality of the infinite variance case is the emergence of a ‘core network’ consisting of nodes with high degrees, through which the short paths typically travel. It is now natural to ask, what happens to these distances, if vertices with highest degrees are deleted. Our main findings can be summarized as follows:

- (i) Any deletion of core vertices leaves the asymptotical size of the largest, giant component intact.
- (ii) If vertices with capacity higher than N^γ are deleted, γ not depending on N , then paths that would have gone through the deleted vertices can be repaired using ‘back-up paths’ that increase the distances only by a constant depending on γ but not on N . This increase is negligible on the $\log \log N$ scaling of distances.
- (iii) If the whole core is removed (for the exact meaning of this, see Section 2), the distances still scale slightly better than $\log N$.

Because of structural similarity (compare [8] with [7]), we conjecture that our results are transferable to the configuration model of Newman *et al.* [6, 8]. Perhaps they can be extended even to preferential attachment models in the finite mean and infinite variance region, since van der Hofstad and Hooghiemstra showed recently that these models share with the previously mentioned models the $\log \log$ -scalability of distances [5].

The next section specifies the graph model and reviews the relevant results of [7]. Some of them are slightly sharpened in Section 3. Section 4 applies results on the diameter of classical random graphs to measure the core network in the horizontal direction. The main result on robustness against core losses is stated and proven in Section 5.

2 Model and earlier results

Throughout the paper, we work with the following conditionally Poissonian power-law random graph model [7]. There are N vertices that possess, respectively, i.i.d. ‘capacities’ $\Lambda_1, \dots, \Lambda_N$ with distribution

$$\mathbb{P}(\Lambda > x) = x^{-\tau+1}, \quad x \in [1, \infty); \quad \tau \in (2, 3).$$

Given $\Lambda = (\Lambda_1, \dots, \Lambda_N)$, each pair of vertices $\{i, j\}$, is connected with E_{ij} edges, where

$$E_{ij} \sim \text{Poisson} \left(\frac{\Lambda_i \Lambda_j}{L_N} \right), \quad L_N = \sum_{j=1}^N \Lambda_j,$$

and the E_{ij} ’s are independent. Loops, i.e. the case $i = j$, are included for principle, although they have no significance in the present study. Given Λ , the degree of vertex i then has the distribution $\text{Poisson}(\Lambda_i)$. The resulting random graph is denoted as G_N .

The following coupling between neighborhood shells and branching processes will play an important role, as it did in [7]. Fix N , and assume that the sequence Λ has been generated and probabilities are now conditioned on it. We shall consider a *marked branching process*, i.e., a branching process where each individual is associated with some element of the mark space $\{1, \dots, N\}$. More specifically, let us define a process $(Z, \mathbf{J}) = (Z_n, (J_{n,i}))$, where Z_n is the size of generation n , and $J_{n,i} \in \{1, \dots, N\}$ is the mark of member i of generation n . We set $Z_0 \equiv 1$ and take $J_{0,1}$ from the uniform distribution $U\{1, \dots, N\}$. The process then proceeds so that an

individual bearing mark i gives for each $j = 1, \dots, N$ birth independently to a $\text{Poisson}(\Lambda_i \Lambda_j / L_N)$ distributed number of j -marked members of the next generation.

On the other hand, we consider the neighborhood sequence around a random vertex of G_N . Take a vertex i_0 from uniform distribution, and define recursively

$$\begin{aligned}\mathcal{N}_0(i_0) &= \{i_0\} \\ \mathcal{N}_{k+1}(i_0) &= \left\{ j \in \left(\bigcup_{l=0}^k \mathcal{N}_l(i_0) \right)^c : j \leftrightarrow \mathcal{N}_k(i_0) \right\}.\end{aligned}$$

The following coupling was proven in [7].

Proposition 2.1 *Let (Z, \mathbf{J}) be the marked branching process defined above. Define a reduced process by proceeding generation by generation, i.e., in the order*

$$J_{0,1}; J_{1,1}, J_{1,2}, \dots, J_{1,Z_1}; J_{2,1}, \dots,$$

and pruning (that is, deleting) from (Z, \mathbf{J}) each individual, whose mark has already appeared before, together with all its descendants (these are considered as not seen in the procedure). Denote the resulting finite process by $(\hat{Z}, \hat{\mathbf{J}})$, and let $\hat{\mathcal{J}}_k$ be the set of the marks in generation k of the reduced process. Then, the sequence of the sets $\hat{\mathcal{J}}_k$ has the same distribution as the sequence \mathcal{N}_k .

Moreover, we observed that the size of each generation and its marks can be generated independently. Denote by q_N the distribution

$$q_N(j) \doteq \frac{\Lambda_j}{L_N}, \quad j = 1, \dots, N, \quad (1)$$

and by π^* the mixed Poisson distribution $\text{Poisson}(\Gamma)$, where Γ is a random variable with distribution

$$\mathbb{P}(\Gamma \in dx) = x \mathbb{P}(\Lambda \in dx) / \mathbb{E}\{\Lambda\}.$$

Obviously, the distribution of $\text{Poisson}(\Lambda_{J(N)})$, $J^{(N)} \sim q_N$, converges weakly to π^* in probability.

Let $J_1^{(N)}, J_2^{(N)}, \dots$ be i.i.d. random variables with distribution q_N , and let $J_0^{(N)}$ be a random variable with uniform distribution on $\{1, \dots, N\}$, independent of the previous ones; we often suppress the superscript $\cdot^{(N)}$. Start now by $(\tilde{Z}_0 = 1, J_0)$, and proceed generationwise by drawing the size of the $n+1$ 'th generation from the distribution

$$\tilde{Z}_{n+1} \sim \text{Poisson} \left(\sum_{i=M_{n-1}+1}^{M_n} \Lambda_{J_i} \right),$$

where $M_{-1} = 0$, $M_n = \sum_0^n \tilde{Z}_k$, and giving then each individual a fresh mark from the sequence $J_{M_n+1}, J_{M_n+2}, \dots$

Proposition 2.2 *The marked generation sequence $(Z_n, (J_{n,i}))$ is stochastically equivalent to*

$$(\tilde{Z}_n, (J_{M_{n-1}+1}, \dots, J_{M_n})).$$

We next turn to the definition of the core network mentioned in the introduction. Fix an increasing function $\ell : \mathbb{N} \rightarrow \mathbb{R}$ such that

$$\ell(1) = 1, \quad \frac{\ell(N)}{\log \log \log N} \rightarrow 0, \quad \frac{\ell(N)}{\log \log \log \log N} \rightarrow \infty \quad \text{as } N \rightarrow \infty, \quad (2)$$

and define

$$\epsilon(N) := \frac{\ell(N)}{\log N}.$$

Note that

$$\epsilon(N) \rightarrow 0, \quad N^{\epsilon(N)} = e^{\ell(N)} \rightarrow \infty \quad \text{as } N \rightarrow \infty. \quad (3)$$

Define recursively the functions

$$\begin{aligned} \beta_0(N) &= \frac{1}{\tau - 1} + \frac{\epsilon(N)}{\tau - 2}, \\ \beta_j(N) &= (\tau - 2)\beta_{j-1}(N) + \epsilon(N), \quad j = 1, 2, \dots, \end{aligned} \quad (4)$$

and denote

$$k^* \doteq \left\lceil \frac{\log \log N}{-\log(\tau - 2)} \right\rceil.$$

For N sufficiently large, we have $\beta_0(N) > \epsilon(N)/(3 - \tau)$, and the sequence $(\beta_k(N))_{k=0,1,\dots}$ decreases toward the limit value $\epsilon(N)/(3 - \tau)$. With $k = k^*$, we are already in the $\epsilon(N)$ order of magnitude:

$$\beta_{k^*}(N) \leq \epsilon(N) \left(1 + \sum_{i=0}^{\infty} (\tau - 2)^i \right) = \frac{4 - \tau}{3 - \tau} \epsilon(N). \quad (5)$$

The key of our analysis of G_N is the notion of its *core* C , consisting of all vertices with capacity higher than $N^{\beta_{k^*}}$. Note that the exact boundary of the core depends on $\ell(N)$ and is thus somewhat arbitrary.

By the definition of $\epsilon(N)$, we can choose a natural number $\kappa = \kappa(N)$ such that

$$\frac{\kappa(N)}{N^{\theta \epsilon(N)}} \rightarrow \infty, \quad \frac{\kappa(N)}{k^*} \rightarrow 0, \quad \text{as } N \rightarrow \infty, \quad (6)$$

where $\theta = (\tau - 2)(4 - \tau)/(3 - \tau)$. We can now collect the main results of [7] to the following theorem:

Theorem 2.3 (i) *The graph G_N has a.a.s. (asymptotically almost surely) a giant component, whose relative size approaches the value*

$$1 - \sum_{j=1}^{\infty} \mathbb{P}(D = j) \mathbb{P}(Z_{\infty}^{(\pi^*)} = 0)^j, \quad (7)$$

where D is distributed as the conditionally Poissonian variable $\text{Poisson}(\Lambda)$, and $(Z_n^{(\pi^*)})$ is a Galton-Watson branching process with offspring distribution π^* .

- (ii) A random vertex of the giant component is connected with the core C in less than $\kappa(N)$ hops. The distribution of the capacity of the first core vertex found in a random-order breadth-first neighborhood search is asymptotically identical to the distribution of $J_{n_C}^{(N)}$, where $n_C = \inf \left\{ n : J_n^{(N)} \in C \right\}$.
- (iii) A random vertex of the core is a.a.s. connected with the vertex i^* with highest capacity in at most k^* hops.
- (iv) As a consequence of the previous items, the distance between two randomly chosen vertices of the giant component is at most $2k^*(N)(1 + o(1))$ a.a.s..

Denote $\mathcal{T}_{\gamma,\delta} = \{i \in \{1, \dots, N\} : \Lambda_i \in (N^\gamma, N^\delta]\}$. The idea of the proof of claim (iii) of Theorem 2.3 is that if the core is divided into ‘tiers’

$$\begin{aligned} V_0 &= \{i^*\}, \\ V_1 &= \mathcal{T}_{\beta_1, \infty} \setminus \{i^*\}, \\ V_k &= \mathcal{T}_{\beta_k, \beta_{k-1}}, \quad k = 2, \dots, k^*, \end{aligned}$$

then, for any $k > 0$, a random vertex of tier k has a.a.s. an edge to $V_{k-1} \cup \dots \cup V_0$.

3 Auxiliary results

In this section we sharpen the estimate used in [7] for the aggregated capacity of a set of form $\{i : \Lambda_i > N^\gamma\}$. In particular, we obtain that the aggregated capacity of a tier $\mathcal{T}_{\gamma, \gamma+\epsilon(N)}$ is asymptotically overwhelmingly larger than all strictly higher vertex capacities together. This holds also when γ depends on N .

We use frequently the notation $x + I := \{x + y : y \in I\}$, where $x \in \mathbb{R}$ and $I \subset \mathbb{R}$ is an interval.

Lemma 3.1 *Let $\alpha = \alpha(N) \in (0, \bar{\alpha})$, where $\bar{\alpha} < 1/(\tau - 1)$. Then, for N sufficiently large,*

$$\mathbb{P} \left(\frac{\sum_{i=1}^N \Lambda_i 1_{\{\Lambda_i > N^\alpha\}}}{N^{1-(\tau-2)\alpha} \mathbb{E} \{\Lambda\}} \in \left(\frac{1}{4}, 4\right) \right) \geq 1 - \left(\frac{\mathbb{E} \{\Lambda\}}{2} \right)^{-\tau+1} N^{-(1-(\tau-1)\bar{\alpha})(\tau-2)}.$$

Proof Denote

$$p(n) = \mathbb{P} \left(\frac{\frac{1}{n} \sum_{i=1}^n \Lambda_i}{\mathbb{E} \{\Lambda\}} \notin \left(\frac{1}{2}, 2\right) \right), \quad N_\alpha = \sum_{i=1}^n 1_{\{\Lambda_i > N^\alpha\}}.$$

Since $\mathbb{P}[\Lambda > y \mid \Lambda > x] = \mathbb{P}(x\Lambda > y)$, we have the distribution equality

$$\sum_{i=1}^N \Lambda_i 1_{\{\Lambda_i > N^\alpha\}} \stackrel{\mathcal{D}}{=} N^\alpha \sum_{k=1}^{N_\alpha} \tilde{\Lambda}_k, \quad (8)$$

where the $\tilde{\Lambda}_k$'s are fresh independent copies of Λ . Then

$$\begin{aligned} & \mathbb{P}\left(\frac{\frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \tilde{\Lambda}_k}{\mathbb{E}\{\Lambda\}} \notin \left[\frac{1}{2}, 2\right]\right) = \mathbb{E}\{p(N_\alpha)\} \\ & \leq \mathbb{P}\left(N_\alpha < \frac{1}{2}\mathbb{E}\{N_\alpha\}\right) + \mathbb{E}\left\{p(N_\alpha) 1_{\{N_\alpha \geq \frac{1}{2}\mathbb{E}\{N_\alpha\}\}}\right\}. \end{aligned} \quad (9)$$

The distribution of N_α is $\text{Bin}(N, N^{-(\tau-1)\alpha})$. The well-known bounds for a binomial random variable X ,

$$\begin{aligned} \mathbb{P}(X < \mathbb{E}\{X\} - x) & \leq \exp\left(-\frac{x^2}{2\mathbb{E}\{X\}}\right), \\ \mathbb{P}(X > \mathbb{E}\{X\} + x) & \leq \exp\left(-\frac{x^2}{2\mathbb{E}\{X\}} + \frac{x^3}{\mathbb{E}\{X\}^3}\right), \end{aligned}$$

yield

$$\mathbb{P}\left(\frac{N_\alpha}{\mathbb{E}\{N_\alpha\}} < \frac{1}{2}\right) \leq e^{-\mathbb{E}\{N_\alpha\}/8}, \quad \mathbb{P}\left(\frac{N_\alpha}{\mathbb{E}\{N_\alpha\}} > 2\right) \leq e^{-\mathbb{E}\{N_\alpha\}/8+1}. \quad (10)$$

By the well-known result on subexponential variables with finite mean (see, e.g., [4]),

$$\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n \Lambda_i > 2\mathbb{E}\{\Lambda\}\right) \sim \mathbb{P}(\max(\Lambda_1, \dots, \Lambda_n) > n\mathbb{E}\{\Lambda\}) \sim \mathbb{E}\{\Lambda\}^{-\tau+1} n^{-\tau+2}.$$

For the large deviation to the opposite direction, the classical Cramér theorem applies and the probability goes to zero at exponential speed. Thus, for N sufficiently large,

$$\mathbb{E}\{p(N_\alpha)\} \leq e^{-\mathbb{E}\{N_\alpha\}/8} + \frac{3}{2}\mathbb{E}\{\Lambda\}^{-\tau+1} \left(\frac{\mathbb{E}\{N_\alpha\}}{2}\right)^{-\tau+2}. \quad (11)$$

Combining (8)-(11), replacing α by the worse case $\bar{\alpha}$ and upperbounding the exponential terms by replacing the factor $3/2$ in (11) by 2 we get the assertion. \square

Lemma 3.2 *Let $\alpha_0(N), \alpha_1(N) \in (0, \bar{\alpha})$, where $\bar{\alpha} < 1/(\tau - 1)$, and $\alpha_0(N) \leq \alpha_1(N) - \epsilon(N)$. Then, for N sufficiently large,*

$$\mathbb{P}\left(\frac{\sum_{i=1}^N \Lambda_i 1_{\{\Lambda_i \in (N^{\alpha_0}, N^{\alpha_1}]\}}}{N^{1-(\tau-2)\alpha_0} \mathbb{E}\{\Lambda\}} \in \left(\frac{1}{5}, 5\right)\right) \geq 1 - 2 \left(\frac{\mathbb{E}\{\Lambda\}}{2}\right)^{-\tau+1} N^{-(1-(\tau-1)\bar{\alpha})(\tau-2)}.$$

Proof This follows by applying Lemma 3.1 to both α_0 and α_1 and noting that $N^{-(\tau-2)(\alpha_1-\alpha_0)} \rightarrow 0$. \square

Lemma 3.3 *For any fixed $b > 1$,*

$$\frac{\sum_{i=1}^N \Lambda_i 1_{\{\Lambda_i > N^{b\epsilon(N)}\}}}{\sum_{i=1}^N \Lambda_i 1_{\{\Lambda_i > N^{\epsilon(N)}\}}} \rightarrow 0 \quad \text{in probability.}$$

Proof By Lemma 3.1,

$$\frac{\sum_{i=1}^N \Lambda_i 1_{\{\Lambda_i > N^{b\epsilon(N)}\}}}{\sum_{i=1}^N \Lambda_i 1_{\{\Lambda_i > N^{\epsilon(N)}\}}} \cdot N^{(b-1)(\tau-2)\epsilon(N)} \in [\frac{1}{16}, 16] \quad \text{a.a.s.,}$$

and the claim follows since $N^{\epsilon(N)} \rightarrow \infty$. \square

Lemma 3.3, combined with our earlier results in [7], entails the important observation that the first contact to the core is a.a.s. close to its bottom:

Proposition 3.4 *Fix $b > 0$. Let I_N be a random node of the giant component of G_N . Asymptotically almost surely, I_N is connected to $\mathcal{T}_{\beta_{k^*}, b\beta_{k^*}}$ with a path that avoids the set $\{i : \Lambda_i > N^{b\beta_{k^*}}\}$, and whose length coincides with the distance between I_N and the core.*

Proof Since the proportional size of the core shrinks to zero, it suffices to consider the case that I_N is picked from outside the core. By the results in [7], conditional that I_N is connected with the core, one of the minimal length paths to the core has its core endpoint distributed as the first core element in an i.i.d. sequence $J_n \in \{1, \dots, N\}$ with common distribution (1). Now, β_{k^*} is proportional to $\epsilon(N)$, and Lemma 3.3 can be applied with β_{k^*} in the role of $\epsilon(N)$. We conclude that the first core element in the sequence J_n belongs a.a.s. to the set $\mathcal{T}_{\beta_{k^*}, b\beta_{k^*}}$, and the statement of the proposition follows. \square

4 Horizontal paths in the core

In this section, we show that quite thin tiers of the form $\mathcal{T}_{\gamma', \gamma''}$ are internally (as induced subgraphs) almost connected in the sense that the proportional size of the largest component approaches one and, moreover, the diameter of that largest component is obtained with high accuracy from remarkable results on classical random graphs. For deterministic γ , the tiers $\mathcal{T}_{\gamma-\epsilon(N), \gamma}$ are a.a.s. connected, and their diameters can be picked almost unequivocally from the following classical theorem [1].

Theorem 4.1 *Consider the Erdős-Rényi random graph $G_{n,p}$ and let $p = p(n)$ and $d = d(n) > 2$ satisfy*

$$\begin{aligned} \frac{\log n}{d} - 3 \log \log n &\rightarrow \infty \\ p^d n^{d-1} - 2 \log n &\rightarrow \infty \\ p^{d-1} n^{d-2} - 2 \log n &\rightarrow -\infty \end{aligned}$$

Then $G_{n,p}$ has diameter d a.a.s.

To include also lower parts of the core, we apply the following, rather recent result by Chung and Lu [2].

Theorem 4.2 *If $np \rightarrow \infty$ and $(\log n)/(\log np) \rightarrow \infty$, then almost surely*

$$\text{diam}(G_{n,p}) = (1 + o(1)) \frac{\log n}{\log np},$$

where $\text{diam}(G)$ denotes the diameter of the largest (giant) component of G .

Denote

$$w(\gamma) := \left\lceil \frac{1 - (\tau - 1)\gamma}{(3 - \tau)\gamma} \right\rceil. \quad (12)$$

Proposition 4.3 *Let $\gamma(N)$ be a non-increasing function such that $\gamma(N) \in [\epsilon(N), \frac{1}{2})$ for all N . If $\lim \gamma(N) > 0$, assume also that $\gamma(\infty)$ does not correspond to a jump of the ceiling function in (12). Then, there exists another function $\gamma_*(N)$ such that*

$$\gamma_*(N) < \gamma(N), \quad \gamma_*(N)/\gamma(N) \rightarrow 1, \quad |\mathcal{T}_{\gamma_*(N), \gamma(N)}| \rightarrow \infty,$$

and

$$\text{diam}(\mathcal{T}_{\gamma_*, \gamma}) = (1 + o(1))w(\gamma(N)) \quad \text{a.a.s.}, \quad (13)$$

where $\text{diam}(\cdot)$ is generally interpreted as in Theorem 4.2. When $\gamma(N)$ is bounded away from zero, $\mathcal{T}_{\gamma_*, \gamma}$ is a.a.s. connected and its diameter is a.a.s. equal to $w(\gamma(N))$.

Proof We shall suppress the N 's in $\gamma(N)$ etc. when it improves the clarity of presentation.

Considered as an induced subgraph, $\mathcal{T}_{\gamma_*, \gamma}$ is denser (resp. sparser) than the classical random graph G_{n, p_*} (resp. G_{n, p^*}) with

$$n = n(N) = |\mathcal{T}_{\gamma_*, \gamma}|, \quad p_* = p_*(N) = \frac{N^{2\gamma_*}}{L_N}, \quad p^* = p^*(N) = \frac{N^{2\gamma}}{L_N}.$$

Denote $\gamma_\delta(N) = \gamma - \delta\epsilon(N)$, where $\delta \in (0, 1)$. We fix δ for a while and choose $\gamma_* = \gamma_\delta$. By Lemma 3.2, $|\mathcal{T}_{\gamma_*, \gamma}| \rightarrow \infty$, and

$$\frac{|\mathcal{T}_{\gamma_*, \gamma}|}{N^{1-(\tau-1)\gamma_*}} \in \left[\frac{1}{5}, 5\right], \quad \text{a.a.s.}$$

On the other hand,

$$\frac{L_N/N}{\mathbb{E}\{\Lambda\}} \in \left[\frac{1}{2}, 2\right] \quad \text{a.a.s.}$$

Thus, a.a.s.,

$$\begin{aligned} \log n &\in (1 - (\tau - 1)\gamma_*) \log N + [-\log 4, \log 4], \\ \log(np_*) &\in (3 - \tau)\gamma_* \log N - \log \mathbb{E}\{\Lambda\} + [-\log 8, \log 8], \\ \log(np^*) &\in (2\gamma - (\tau - 1)\gamma_*) \log N - \log \mathbb{E}\{\Lambda\} + [-\log 8, \log 8]. \end{aligned}$$

Note that $p^*(N) \rightarrow 0$ and $n(N)p_*(N) \rightarrow \infty$, so Theorem 4.2 applies to G_{n,p_*} and G_{n,p^*} . We distinguish between two cases.

First, if $\gamma(N)$ is bounded away from zero, we are in fact in the regime of Theorem 4.1, and a computation of the diameter d leads to the expression of $w(\gamma)$. Thus, Theorem 4.1 yields the inequalities

$$\begin{aligned} \text{diam}(\mathcal{T}_{\gamma_*, \gamma}) &\leq \left\lceil \frac{1 - (\tau - 1)\gamma_*}{(3 - \tau)\gamma_*} \right\rceil \quad \text{a.a.s.}, \\ \text{diam}(\mathcal{T}_{\gamma_*, \gamma}) &\geq \left\lfloor \frac{1 - (\tau - 1)\gamma_*}{2\gamma + (1 - \tau)\gamma_*} \right\rfloor \quad \text{a.a.s.} \end{aligned}$$

Now, the right hand sides are with sufficiently large N both equal to the right hand side of $w(\gamma)$, and the claim follows. The value of δ did not matter.

The second case is that $\gamma(N) \searrow 0$. Since $n(N)p_*(N) \rightarrow \infty$, the subgraph still has a giant component whose proportional size approaches one, although it is not necessarily connected. Choose any sequence $\delta_k \searrow 0$, and denote

$$\eta_k = \max \left\{ \frac{(3 - \tau)\gamma}{(3 - \tau)\gamma_{\delta_k}}, \frac{2\gamma + (\tau - 1)\gamma_{\delta_k}}{(3 - \tau)\gamma} \right\}.$$

Note that $\eta_k \rightarrow 1$. Now define a sequence N_k as

$$\begin{aligned} N_k &= \inf \left\{ M : \mathbb{P} \left(|\mathcal{T}_{\gamma_{\delta_k}, \gamma}| \frac{N^{2\gamma_{\delta_k}}}{L_N} \geq k \right) \geq 1 - 1/k \quad \forall N \geq M, \right. \\ &\quad \left. \mathbb{P} \left(\text{diam}(\mathcal{T}_{\gamma_*, \gamma}) \in (\eta_k^{-2}, \eta_k^2) \cdot \frac{1}{(3 - \tau)\gamma} \right) \geq 1 - 1/k \quad \forall N \geq M \right\}. \end{aligned}$$

Thanks to Theorem 4.2, the N_k 's are finite numbers, and obviously they grow unboundedly. Finally, denote

$$k(N) := \max \{k : N_k \leq N\}$$

and choose $\gamma_*(N) = \gamma_{\delta_{k(N)}}$. Then obviously

$$\text{diam}(\mathcal{T}_{\gamma_*, \gamma}) = (1 + o(1)) \frac{1}{(3 - \tau)\gamma(N)} \quad \text{a.a.s.}$$

□

In a similar way, we can prove a diameter result for the tiers $V_k = V_k^{(N)}$, defined at the end of Section 2:

Proposition 4.4 *The tiers V_0, \dots, V_{k^*} , considered as induced subgraphs of G_N , are almost connected in the sense that the relative sizes of their largest components approach 1 as $N \rightarrow \infty$, and their diameters satisfy, a.a.s.,*

$$\begin{aligned} \text{diam}(V_0) &= 0, \\ \text{diam}(V_1) &= 2, \\ \text{diam}(V_k) &\in (1 + o(1))[w(\beta_{k-1}), w(\beta_k)], \quad k = 2, \dots, k^*. \end{aligned}$$

By Proposition 4.3, we can say that the width of the core at height N^γ , measured by hop distance, is $w(\gamma)$. Moreover, this holds also for varying $\gamma = \gamma(N)$ down to the bottom of the core.

When $\gamma \in (0, \frac{1}{2})$, we have

$$w((\tau - 2)\gamma) > w(\gamma) + 2.$$

This has the following heuristic consequence. For connecting two vertices with capacities $\approx N^\gamma$, a ‘horizontal’ path staying at about the same height in the core is longer than a path that jumps at both ends with one hop to the height $N^{(\tau-2)\gamma}$ and finds the horizontal connection at that level. On the other hand, this is how high in the core the neighborhood of a vertex typically reaches. Thus, a horizontal move should be made at as high level as possible.

5 The main result

We have now essentially collected all elements needed to understand the situation where a top part of the core is deleted. First, since the aggregated capacity of any top part is concentrated at its bottom, the remains of the core are found from outside almost as easily as the original core was found. Second, if a lower part of the core is alive, we can find ‘vertical’ paths from the bottom of the core to a top tier of its remaining part, as we did with the undamaged core in [7] (a complete proof of this step requires a slight modification of the proof of Proposition 4.3 of [7] and is presented below in detail). Third, Proposition 4.3 guarantees the existence of a ‘horizontal’ path between almost any pair of vertices within this top tier, and provides also an explicit upper bound for its length. It remains to couple these pieces together.

Theorem 5.1 *Let $\gamma := \gamma(N) \in (\epsilon(N), \frac{1}{2})$ be such that $\gamma(N)$ is non-increasing and $\gamma(N)/\epsilon(N)$ non-decreasing w.r.t. N . Denote by $H_\gamma = H^{(N)}$ the graph obtained from G_N by deleting all vertices with capacity higher than N^γ , together with the edges attached to them. The following hold:*

- (i) *The graph H_γ has a.a.s. a giant component whose relative size approaches the relative size of the giant component of G_N .*
- (ii) *The distance of two randomly chosen vertices of the giant component of H_γ is a.a.s. less than*

$$(1 + o(1)) \left(\frac{2}{-\log(\tau - 2)} \left(\log \log N - \log \frac{1}{\gamma} \right) + w(\gamma) \right), \quad (14)$$

where $w(\gamma)$ is given in (12).

Proof 1°. Denote

$$\begin{aligned} \gamma_0 &= \gamma, \\ \gamma_1 &= \gamma_0 - \epsilon(N), \\ \gamma_{k+1} &= (\tau - 2)\gamma_k + \epsilon(N), \quad k = 1, 2, \dots, \end{aligned}$$

and let

$$\bar{k} = \min \left\{ k \geq 1 : \gamma_k \leq \frac{4-\tau}{3-\tau} \epsilon(N) \right\}.$$

It is easy to check (see 4° below) that $\bar{k} = O(\log \log N)$.

Consider the tiers $\mathcal{T}_{\gamma_1, \gamma_0}, \mathcal{T}_{\gamma_2, \gamma_1}, \dots, \mathcal{T}_{\gamma_{\bar{k}}, \gamma_{\bar{k}-1}}$. The idea of the proof is to apply the diameter result, Theorem 4.2, on $\mathcal{T}_{\gamma_1, \gamma_0}$, and, if $\bar{k} \geq 2$, to imitate our proof of Theorem 2.3 to show that vertices of tier $\mathcal{T}_{\gamma_{\bar{k}}, \gamma_{\bar{k}-1}}$ are with high probability connected to $\mathcal{T}_{\gamma_1, \gamma_0}$ with a path jumping from tier to tier.

2°. Assume that $\bar{k} \geq 2$. Let $I_0 = I_0(N)$ be a stochastic (not necessarily uniformly random) vertex of $\mathcal{T}_{\gamma_{\bar{k}}, \gamma_{\bar{k}-1}}$ such that, conditioned on Λ , I_0 is independent of the edges within $\bigcup_1^{\bar{k}} \mathcal{T}_{\gamma_j, \gamma_{j-1}}$. Define the sequence of vertices I_1, I_2, \dots recursively as follows. If $I_n \in \mathcal{T}_{\gamma_{\bar{k}-n}, \gamma_{\bar{k}-n-1}}$, then I_{n+1} is a neighbor (say, with smallest index) of I_n belonging to $\mathcal{T}_{\gamma_{\bar{k}-n-1}, \gamma_{\bar{k}-n-2}}$ if such a neighbor exists. Otherwise $I_{n+1} = I_n$, and the rest of the sequence repeats I_n as well. Denote

$$\begin{aligned} A_n &= \{I_n \in \mathcal{T}_{\gamma_{\bar{k}-n}, \gamma_{\bar{k}-n-1}}\}, \quad n = 0, 1, \dots, \bar{k} - 1, \\ B' &= \left\{ \frac{L_N}{N} \leq 2\mathbb{E}\{\Lambda\} \right\}, \\ B_k &= \left\{ L(\mathcal{T}_{\gamma_{k+1}, \gamma_k}) \in \mathbb{E}\{\Lambda\} N^{1-(\tau-2)\gamma_k} \left(\frac{1}{5}, 5\right) \right\}, \quad k = 0, \dots, \bar{k} - 1, \\ &\text{where } L(S) := \sum_{i \in S} \Lambda_i, \\ B &= B' \cap B_0 \cap \dots \cap B_{\bar{k}-1}. \end{aligned}$$

We have $A_{\bar{k}-1} \subseteq A_{\bar{k}-2} \subseteq \dots \subseteq A_0$. Proposition 3.2 yields

$$\mathbb{P}(B_0 \cap \dots \cap B_{\bar{k}-1}) \geq 1 - 2\bar{k} \left(\frac{\mathbb{E}\{\Lambda\}}{2} \right)^{-\tau+1} N^{-(1-(\tau-1)\gamma)(\tau-2)} \rightarrow 1$$

as $N \rightarrow \infty$. On B in turn, we have for each n by Proposition 3.2 that

$$\begin{aligned} \mathbb{P}[A_{n+1}^c \mid \Lambda, A_n] &= \mathbb{E} \left[\exp \left(-\frac{\Lambda_{I_n} L(\mathcal{T}_{\gamma_{\bar{k}-n-1}, \gamma_{\bar{k}-n-2}})}{L_N} \right) \mid \Lambda, A_n \right] \\ &\leq \exp \left(-\frac{1}{2\mathbb{E}\{\Lambda\} N} N^{\gamma_{\bar{k}-n}} \cdot \frac{1}{5} N^{1-(\tau-2)\gamma_{\bar{k}-n-1}} \right) \\ &\leq \exp \left(-\frac{e^{\ell(N)}}{10\mathbb{E}\{\Lambda\}} \right). \end{aligned}$$

It follows that on B

$$\begin{aligned} \mathbb{P}[A_{\bar{k}-1} \mid \Lambda, A_0] &= \prod_{n=0}^{\bar{k}-1} \mathbb{P}[A_{n+1}^c \mid \Lambda, A_n] \\ &\geq \left(1 - \exp \left(-\frac{e^{\ell(N)}}{10\mathbb{E}\{\Lambda\}} \right) \right)^{\bar{k}} \end{aligned}$$

$$\begin{aligned}
&\geq 1 - \bar{k} \exp(-c e^{\ell(N)}) \\
&= 1 - \frac{\bar{k}}{\log^2 N} \exp\left(\log^3 N - c(e^{\log^4 N})^{\ell(N)/\log^4 N}\right) \\
&= 1 - \frac{\bar{k}}{\log^2 N} \exp\left(-\log^3 N \left(c(\log^3 N)^{\ell(N)/\log^4 N - 1} - 1\right)\right) \\
&\rightarrow 1 \quad \text{as } N \rightarrow \infty
\end{aligned}$$

by the assumption that $\ell(N)/\log^4 N \rightarrow \infty$.

3°. Now let I and I' be two independent uniformly randomly chosen vertices of G_N . By Proposition 3.4, I is a.s. either outside the largest component of G_N , or it is connected to $\mathcal{T}_{\gamma_{\bar{k}}, \gamma_{\bar{k}-1}}$ with a path that avoids the set $\{i \leq N : \Lambda_i > N^{\gamma_{\bar{k}-1}}\}$ and whose length is at most $\kappa(N)$. Denote the endpoint of that path by I_0 . In the case that $\bar{k} \geq 2$, step 2° above showed that I_0 is a.s. connected to a vertex $I_{\bar{k}-1} \in \mathcal{T}_{\gamma_1, \gamma_0}$ over a path of length \bar{k} . The same claims holds for I' , with vertices I'_0 and $I'_{\bar{k}-1}$ respectively. Now, $I_{\bar{k}-1}$ and $I'_{\bar{k}-1}$ belong a.s. to the largest component of the induced subgraph $\mathcal{T}_{\gamma_1, \gamma_0}$, and Theorem 4.2 yields that their distance is at most $(1 + o(1))w(\gamma)$.

4°. We have now shown that if I and I' both belong to the largest component of G_N , their distance is a.s. at most

$$(1 + o(1))(2\bar{k} + w(\gamma)).$$

It remains to show that \bar{k} can in this expression be replaced by

$$\frac{\log \log N - \log(1/\gamma)}{-\log(\tau - 2)}. \quad (15)$$

First, if $\gamma(N) \leq C\epsilon(N)$ for all N and some finite $C > 1$, then

$$\log \frac{1}{\gamma} \in (\log \log N - \log \ell(N)) + [-\log C, 0],$$

and (15) is in this region negligible compared with $w(\gamma)$. On the other hand, \bar{k} is in this region limited by a constant, whereas $w(\gamma(N)) \rightarrow \infty$. Thus, the assertion of the theorem holds.

Assume then that $\gamma(N)/\epsilon(N)$ is unbounded. Then $\gamma_1 > ((4 - \tau)/(3 - \tau))\epsilon(N)$ for large N , so that $\bar{k} \geq 2$. Since

$$\gamma_{k+1} = (\tau - 2)^k \gamma_1 + \epsilon(N) \sum_{i=0}^{k-1} (\tau - 2)^i,$$

a short computation (using the geometric series formula) yields that the condition $\gamma_k \leq ((4 - \tau)/(3 - \tau))\epsilon(N)$ is equivalent to the condition

$$(\tau - 2)^k \left(\gamma_1 - \frac{\epsilon(N)}{3 - \tau} \right) \leq \epsilon(N),$$

and further to

$$\begin{aligned}
k &\geq \frac{1}{-\log(\tau - 2)} \left(\log \log N + \log \left(\gamma(N) - \frac{4 - \tau}{3 - \tau} \epsilon(N) \right) - \log \ell(N) \right) \\
&= (1 + o(1)) \frac{\log \log N - \log(1/\gamma(N))}{-\log(\tau - 2)}.
\end{aligned}$$

Thus, \bar{k} equals the last, asymptotic expression. This finishes the proof. \square

Theorem 5.1 tells that removing a top part of the core down to N^γ with fixed $\gamma > 0$ has no effect to the asymptotic upper bound (14). On the other hand, with $\gamma(N) \propto \epsilon(N)$ we obtain

$$w(\gamma(N)) \propto \frac{1}{(3 - \tau)\epsilon(N)} = \frac{\log N}{(3 - \tau)\ell(N)}.$$

Combining this with Proposition 3.4, we find that if the whole core is removed, the remaining graph still has a giant component with same proportional size as the original and diameter proportional to $\log(N)/\ell(N)$ — slightly smaller than that of a (supercritical) power-law graph with $\tau > 3$. Asymptotics of intermediate cases between these two extremes can be computed from (3.4) as well.

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